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STUDIES IN THE STRUCTURE OF SYSTEMS

3. POSTULATES

THE structural elements of the classical theory of the deductive system are: *proof, definition, categories, axioms, theorems.* Whilst all these are integral elements of the deductive system, it is "proof," or "deduction," in terms of which the form is defined. A systematic account might therefore be expected to begin with "proof." I find it, however, more convenient to introduce my exposition with a critical account of the main propositions regarding "axioms," because it is through the more patent changes here that the required changes in our theory of "proof" will become evident.

It is interesting that neither the term nor the usual meaning of "axiom" has any place in Plato's philosophy. It is characteristically Aristotelian. Though the Stagirite's theory of the deductive system rests on the basis laid by Socrates—Plato (a fact which is merely obscured by his numerous criticisms of Plato's doctrine), regarding "axioms," he fundamentally differed from the latter. Plato's term is *ὑπόϑεσις*, which in its meaning and function corresponds to the modern use of "postulate."

Aristotle's theory of the deductive system is dominated by his conception of "cognition" (*ἐπιστήμη*). Cognition is "necessary," "certain," "apodictical" and implies the idea of a "cause" (*αἰτία*) on which it rests and from which it follows with necessity.[1] A proof (*ἀπόδειξις*) is a syllogism which leads to cognition (*συλλογιςμός ἐπιστημόνικος*), and from the point of view of "proving," the syllogism was elaborated by Aristotle as the methodical procedure which determines the "following with necessity." In every proof Aristotle distinguishes three parts: first, that which is proved, the conclusion (*τὸ ἀποδεικνύμενον τὸ συμπέρασμα*); secondly, the *axioms,* from which (*ἐξ ὧν*) the proof proceeds; and, thirdly, the subject whose properties the proof exhibits (*τὸ γένος τὸ ὑποκαίμενον*).[2] But the proof, this mediator of cognition, making the truth of one proposition rest on that of another, can neither move in a circle nor regress indefinitely; it must come to a standstill; there must be a "first" of cognition: secondly, we say: "not only is there cognition, but also a first of cognition (*ἀρχὴ ἐπιστήμης*)."[3] The *ἀρχὴ* is twofold, namely, *ἐξ ὧν* and *περὶ ὅ*.[4] The latter comprises the special presuppositions in each proof, the *δαιΐι*; they are usually interpreted as meaning the "special principles of each science." This interpretation is indeed suggested by the end of Chapter XXXII., referred to above, as well as by the remarkable passage at the beginning of Chapter VII (which anticipates what we are pleased to call the modern insistence on the "purity of methods"). But this interpretation, though supported by the authority of Zeller, is certainly too narrow, for even these "special principles" are "general"; but each proof requires the "particular"; and Aristotle insists that the number of "principles" (*ἀρχὴ*) is not much smaller than that of the "conclusions," and can not be finite if that of the conclusions is infinite.[5] The *ἴδιαι* are therefore not merely the special principles of each "science," but of each *ἀποδευκνύμενον τὸ συμπέρασμα*: wherever there is proof it proceeds, in part, *ἐκ τῶν ἑκάστου ἀρχῶν*. Such is the *περὶ ὅ*, such *τὸ γένος τὸ ὑποκαίμενον*. "Number," "magnitude" are mentioned as examples; it is the *γένος*, the concept, and we shall have to consider this part of the *ἀρχὴ* in a later paper.

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[1] Analyt. post. I_2.

[2] Analyt. post. I_7.

[3] *Ibid.,* Chap. 3.

[4] *Ibid.,* Chap. 32.

The "axioms," on the other hand, are the *ἐξ ὧν*, the *κοίναι*, as opposed to the *ἴδιαι*, in each proof. As examples of "axioms" are mentioned the principle of contradiction, of excluded middle, that equals subtracted from equals are equal.

What is the nature of the *ἀρχὴ* on which all cognition rests? Aristotle concludes that it must be *true*, the *first, immediate, better known, earlier*, and in *causal relation to the conclusion*. These determinations seem indeed necessary from the Aristotelian point of view. Cognition appears like a building which needs must rest on a secure foundation. And Aristotle has given the reasons for each one of these properties of the *ἀρχή*. It will be necessary to examine them.

The first demands "truth" for the *ἀρχὴ*, a truth which does in no way depend on that which follows from it, neither does itself follow from any other truth; truth belongs to the axiom as such, apart from anything else; the *truth* of the *ἀρχὴ* is *isolated*, even though the *ἀρχὴ* itself stands in relation to other propositions. And it is recognized "immediately"; it is self-evident, for we know it "in still higher degree" than any of the derived propositions. If any proposition is presented, and it is an *ἀρχὴ*, we must be able to decide by direct inspection whether it is true or not; and *vice versa*.

After what has been said in the first paper, it will be apparent that these arguments move in the realm of "psychology of cognition" and of "critique of cognition"; for they deal with problems concerning the *truth* and the *subject-relation* of logical entities. But it was necessary to inspect them here, because they serve, in their part, to establish the *radical distinction* between the *ἀρχὴ* and the "theorems." If Aristotle adds therefore that the "axioms" must be a “first,” it is clear that he refers, not to any accidental order of presentation in a given system, but to an *objective* order, to a relation between the logical entities themselves. The “first” is not merely “unproved,” but *“indemonstrable”* (ἐκ πρώτων δ' ἀναποδείκτων) *i. e.,* that which *can not* be proved; and which need not be proved, because its certainty is superior to all proof.

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[5] *Loc. cit.*

The determinations “better known” and “earlier” seem at first sight to refer merely to a subject-relation; but Aristotle makes the distinction between the πρότερον πρός ἡμᾶς, and the πρότερον τῇ φύσει ; it is the latter he means here, and it stands merely for *“generality”;* for, according to him, “better known and earlier in itself” is that which is further away from sense-perception; and “the most general is furthest away.” Aristotle applies here to propositions a distinction, which is of fundamental importance in his theory of concepts, namely, with respect to “generality”; he seems to assume that, given two propositions p_1 and p_2, they always have a definite relation of generality. And yet it is by no means self-evident that this assumption must hold; on the contrary, it may be doubted whether it really does hold. Aristotle had, no doubt, in his mind examples of syllogisms of the “all men are mortal” type; and here the distinction is simple: “all men are mortal” is more general than “Socrates is mortal.” It was a dogma of the traditional logic that all proofs in mathematics are of this type. Kant took exception to this dogma; Sigwart tried to formulate a difference in his *“Logik”;* but most clearly the difference is exhibited by the modern work in the algebra of relatives. Now take Sigwart's example: “if the corresponding sides of two triangles are in proportion, the corresponding angles are equal; if the corresponding angles of two triangles are equal, the triangles are similar; therefore, if the sides of two triangles are in proportion, the triangles are similar.” This, as Sigwart rightly remarks, looks like Barbara, but is, in reality, *himmelweit davon verschreden.* The propositions do not simply state “subsumptions,” but relations of a different kind. *But which of these propositions is the most general?* In a similar way, which proposition is more general: Euclid's parallel axiom, or the theorem about the sum of the angles in a triangle?

This criticism of the concept of “generality,” as applied to propositions, merely introduces the critique of “axioms,” which the modern work necessitates. Generality might be surrendered; it would not make Aristotle's account any more acceptable. The difference between the ἀρχή and any theorem would still be radical, absolute, inherent; the proof, according to Aristotle, *can* proceed in but one direction: *from* certain propositions (the ἀρχή), *to* certain others: *it never can go in the opposite direction,* at least not as a deductive proof; it is impossible, if *A* is used as a basis for proving *B*, that *B* should be used to prove *A*. We can, from "all men are mortal," deduce "Socrates is mortal," but not inversely. This inverse procedure, prohibited by the Aristotelian theory of a deductive system, we can, however, easily exhibit in mathematics, at least in a large class of cases. Thus if, in plane Euclidean geometry, the opposite sides of a quadrilateral are parallel, it can be proved that they are equal; and *vice versa*. This may not seem very remarkable; but it applies to the *"axioms."* By means of the "parallel axiom" we can prove the theorem about the sum of the angles in a triangle; and *vice versa*. Either one can therefore be "proved," provided the other is accepted. "Indemonstrable" is not a property which inheres in a proposition as such, as Aristotle claimed, but in a proposition *in a system, i. e.*, in relation to others. Take it out of this definite systematic arrangement with its definite *order*, and the term "indemonstrable" becomes meaningless.

Mathematicians have therefore more and more avoided the term "axiom," and speak of "postulates," or "hypotheses," to express that the starting-point of the system is merely "assumed," that the "fundamental propositions" of a system are merely "unproved," but might be "provable" propositions in a different arrangement. Which propositions are chosen as postulates and which as theorems is *accidental*, namely, to the particular arrangement; it is not a logical property of certain propositions to be "presupposed" by others.

This states somewhat radically what has been demonstrated, however, thus far only in part. Mathematicians have worked out numerous "sets of postulates," which exhibit this interchange between postulates and theorems. Is there any limit to this interchangeability? Are there any propositions which always *must* be among the postulates? The views regarding this *degree* of interchangeability differ somewhat; some, *e. g.*, Bertrand Russell, hold that it is possible only within certain (though as yet undefined) limits; others, as E. V. Huntington, incline to the view that this interchange can go on indefinitely; and I myself incline to the latter view; the presumption, at least so far as mathematics is concerned, is strongly in favor of it.

But, some writers hold, whilst this may be true for mathematics, it is not true for logic; such propositions as the "syllogism" *must* be among the "fundamental principles." Two arguments are advanced to support this view.

First, it is held, these propositions are "absolutely true," in the sense that they can not possibly be denied; or, as Professor Royce puts it, their denial implies their own assertion. It would, therefore, be absurd to begin with any other "postulates" if these "sure" and "undeniable" propositions are at our disposal. This view is closely allied to the Aristotelian; it differs from it in that it proposes a definite *logical criterion* for testing the absolute truth of a proposition; it does not rely on psychological "self-evidence"; the principles which are thus declared "absolutely true" by the criterion "that which is implied by its own denial is absolutely true" may indeed be very much lacking in psychological self-evidence.

The second argument, used by Russell and others, is that the very nature of proof demands certain propositions, such as the "principle of the syllogism," that they must therefore be amongst the "postulates," for no proof is possible without them.

I can not consider the first argument here, as it belongs to "critique of cognition"; I have treated it in a paper on "A Class of Invalid Criteria of Truth," in which I believe to have shown that "absolute truth" in any such sense as the argument understands it, does not exist. *Truth is always relative to its particular problem.*

The second argument, however, contains the recognition of an important principle, though it unduly restricts it to such postulates as the "principle of the syllogism." We may state it thus: any deductive system of logic must have among its "postulates" the "principle of the syllogism," *or its equivalent.* But this holds true of all postulates and for all deductive systems. In other words: we do not interchange propositions at haphazard in selecting new sets of postulates from the propositions of the system. *We choose equivalent sets.* But "equivalent" propositions may in all other respects be widely *different;* they are by no means necessarily *identical.* I shall revert to this question of equivalence, which seems of very great importance, in a later paper. For the present purpose it is sufficient to say that equivalence *regulates* but does not *limit* the interchange of propositions in a system. (Incidentally it may be remarked that the "principle of the syllogism" is *not* among *Whitehead's* set of postulates for the algebra of logic, but is *proved* as a *theorem.*)[6] Russell's own position seems to have undergone a change regarding the question of "axioms" and in the direction toward the position taken in this paper. As evidence I quote merely two passages from the "Principles of Mathematics," published in 1903, and their amendments in the "Principia Mathematica" published, in conjunction with Whitehead, in 1910. The passages are: "A definition of implication is quite impossible" (p. 14), supporting Peano's view regarding the existence of "indefinables"; but "Principia Mathematica" *does* define implication! The other quotation is: "Some indemonstrables there must be; and some propositions, such as the syllogism, must be of the number, since no demonstration is possible without them.''[7] In the same connection Russel upholds the distinction between ''unproved'' and ''indemonstrable'' propositions, and explicitly asserts the existence of ''indemonstrables.'' But ''Principia Mathematica'' is less explicit on the point and seems to me in perfect harmony with the view defended in the present paper. ''Thus deduction depends upon the relation of implication, and every deductive system must contain among its premises as many of the properties of implication as are necessary to legitimate the ordinary procedure of deduction'' (p. 94). Only why not frankly say that logic could be developed altogether without even mentioning implication? Mrs. Franklin has developed a system which takes ''inconsistency'' as its fundamental concept; Professor Royce, by means of his 0-relation, is able to dispense with ''implication'' as a fundamental concept; Whitehead did not require it. In all these cases it was easy to ''define'' implication, and ''prove'' the laws of the syllogism.

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[6] Whitehead, "Universal Algebra"; or E. V. Huntington's paper, "Sets of Independent Postulates for the Algebra of Logic," in the *Transactions of the American Mathematical Society,* July, 1904.

If this position, that there are no ''indemonstrables,'' is accepted as the outcome of the modern work, an enormous freedom is gained for logic as well as for mathematics. For then only are we able to rid ourselves successfully of the confusion of purely logical with psychological questions. Ever since Kant has the attempt been made to separate these; but ''logical'' necessity was ever so closely allied with a purely psychological ''not-being-able-to think otherwise'' that a confusion was unavoidable.

But, whilst it is demonstrated that an interchange between ''postulates'' and ''theorems'' is possible, and whilst it is at least probable that this interchange has no logical limitations, it is still possible, and necessary, to *choose*, from the various possible sets of postulates, certain ones as *preferable*, provided the criteria of preference are stated or indicated. And if we look back to Aristotle's theory, we can say: he stated as principle of selection prominently this: ''Choose a set of postulates such that it contains the *most general* propositions of the system.'' And our main objection to Aristotle's theory may now be stated thus: he incorporated his principle of selection into his theory of the *structure* of a deductive system, basing on it an *absolute* distinction between ''fundamental principles'' and ''theorems,'' and denying the possibility of *other* principles of selection. And in this he was wrong. It is not necessary to always prefer the ''most general'' principles. And it may be well to elucidate this by referring to *other* possible principles of selection which have played an important rôle in the history of science. Thus it is interesting to notice that if philosophers have been dominated in their procedure by "generality," mathematicians have striven for "simplicity" of their starting-point. In "geometry" they do not begin with the general properties of curves, but with postulates about straight lines and points; in "algebra" they start with the properties of the "natural" numbers, before developing the properties of "complex quantities"; propositions are not taken in their most general meaning at first, but proved for a limited field and gradually "extended" by proper methods of procedure. In all these it is the "simpler" from which mathematicians start, the more "general" they may strive to reach afterwards. And this procedure is not dictated by any logical necessity; it could be, and sometimes is, inverted. Instead of starting, as Weierstrass did, with the simple properties of "power series" and extending these gradually so as to reach the more general functions, we may, with Riemann, begin by studying the general properties of functions and "apply" these to "algebraic," etc., functions. But in the two cases we obtain a different system of the "Theory of Functions."

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[7] ''Principles of Mathematics,'' page 15.

And may we not, in the selection of sets of postulates, base our preference on purely psychological grounds, such as "evidence"? Why not, indeed? We may admit that "evidence" is not a logical property of propositions, but depends also on the "subject" and his natural surroundings; and what we suppose will be "evident" to our hearers may be far from it! Nevertheless, if a teacher should choose to present a subject in the deductive system form, he would be likely to start, as best he could, with postulates which, to the pupils, have at least a certain degree of "evidence" or even "familiarity." He would not be likely to start a study of "arithmetic" on the basis of Dedekind's *Was sind und was sollen die Zahlen;* he would not choose many of the "primitive propositions" of the "Principia Mathematica"! To Aristotle the "general" *was* the "better known," the deductive procedure the best for *arriving* at cognition. We, who know better than Aristotle possibly could have, how much is really demanded by "logical rigor" may well doubt whether any subject should ever be first presented in a purely deductive form, and may be profoundly thankful that our mathematical school-books fall so palpably short of their much-boasted rigor.

From here the real necessity of a "critique of cognition" will become apparent. If a system, such as geometry, can properly be presented in but one form, we may need a code of rules to detect errors in reasoning. Beyond this, what demand is there for criteria to determine the *logical value* of a given system? If, however, the same content can be presented in many, perhaps an infinity of different forms, all logically faultless, we are put before the questions: how shall we select, and by what ideals shall we be guided in our selection? What do we even mean by saying: this system is *true?* For now we can no longer answer: because it follows from absolutely true and self-evident propositions; the theorems of one are the postulates of another, and this simple transference has not suddenly increased their truth value from the "to be proved" to the "undeniable"! It is curious to notice the attitude of some mathematical philosophers in this respect. They have always insisted on "proofs." And this demand could easily be justified so long as deductive systems were conceived to start from "axioms." Theorems, by their proofs, were made to participate in this "self-evidence" of the axioms. But how, if it is admitted that the starting-point is merely "postulated"? "In mathematics," say the authors of the "Principia Mathematica," "the greatest degree of self-evidence is usually not to be found quite at the beginning, but at some later point; hence the early deductions, until they reach this point, give reasons rather for believing the premises because true consequences follow from them, than for believing the consequences because they follow from the premises" (preface).

This certainly turns the Aristotelian conception of a deductive system upside down: the "postulates" borrow now their certainty from that of the "theorems"; for the latter are *"facts."* It is easily seen how this view, which is shared by others, could have been suggested by the recent work in mathematics. The "theorems" have remained much more invariant than the "postulates." There are numerous sets of postulates for Euclidean plane geometry; but they all leave the Pythagorean proposition "true"; and their own "truth" is, in part, tested by the criterion that this theorem *can* be deduced from them. However, calling a theorem a "fact" does not in any way show how this claim to certainty and truth is warranted. It does not usually mean that the propositions of mathematics are "empirical"; and it leaves open the possibility that the theorem does *not* hold if the "postulates" are suitably changed; *i. e.,* it leaves the possibility of other geometries in which the contradictory opposite of the particular theorem is also a "fact." The term "fact" is therefore used here merely to designate that which is "true" in a given system: it explains nothing, it warrants nothing; in particular, the implication of the ordinary use of the term "fact," namely, that it "holds" quite apart from the truth of anything else, is certainly not meant by those who use the term "fact" here.

It is characteristic of the difference between the attitudes of philosophers and of mathematicians that the question of "certainty," "undeniability" plays so large a rôle with the former, and so small a one with the latter. Through the whole history of philosophy runs this endeavor to find premises which will silence every possible *doubt;* however slim the foundation, however insignificant in itself—a mere *cogito ergo sum,* perhaps—if only it is secure, if only it may serve to refute the radical skeptic! This timidity, this absorbing desire for security, this willingness to sacrifice everything to the feeling: here at last is a proposition that nobody can deny! would be ludicrous were it not so pathetic; for this abstemious self-denial was never rewarded. No sooner had one philosopher retired to his one lone rock, when another began to show that it was a mere drift! Is it not time to recognize that this whole procedure is vain; can philosophers not resign themselves to admit that the radical skeptic's position is impregnable—but also absolutely barren, and that there is no need whatever to *take* or even to invest this stronghold?

Mathematics has always passed as the paragon of security, of undeniability; how envious it must have made some philosophers! Yet, the fear of a possible skeptic has never been one of the obsessions of mathematicians. It is astonishing how often even legitimate objections were simply disregarded—until the proper time had come for their disposal. When the infinitesimal calculus was first invented, it rested on a foundation by no means irreproachable from a logical point of view. But Bishop Berkeley made small headway with his attacks on calculus. It was an efficient instrument in solving problems, which could not be solved by any other method—that was sufficient reason for keeping it. Had a logically superior and practically as efficient an instrument been offered to the mathematician, he would have been quick in discarding calculus. As it was, it was used constructively for over a hundred years before serious attempts were made to improve the logical foundations; more justly it ought to be said: before mathematicians were in a position to improve the foundations. Had they, however, discarded calculus in the first place on the ground of logical imperfections, they would never have been able to find the better foundations. All this is not said in justification of the mathematician's procedure; but it illustrates the difference; mathematicians are supremely interested in "construction": what will follow from certain data; whilst philosophers have concerned themselves primarily with the question: which premises are sufficiently secure to build upon; and this desire has led them to "self-evident" axioms, to "absolute" truth, and now to the assertion that the propositions of mathematics are "facts."

We are not concerned here with the problem of truth; we require merely that no theory of truth should be held which makes certain questions of structure impossible of solution. From the point of view of structure, the radical and inherent distinction between theorems and postulates must be denied. They are logically on a par. The decision—which propositions are chosen for postulates and which proved as theorems—is determined by secondary considerations, prominent amongst which are those of "critique of cognition."

This denial of the radical and inherent distinction between theorems and postulates does, however, not imply the denial of the existence of "order" in a deductive system. *Which* propositions of a given system are chosen as postulates is logically irrelevant, but *that* certain ones be chosen is necessary if a deductive system is to result. In a given deductive system the difference between postulates and theorems is definite: the latter must be proved, the former are "premises" of these "proofs," themselves "unproved" and "indemonstrable," namely, in the given system. To say that the distinction is not radical means therefore merely that the same logical content could have been put into the deductive system form with a different selection of propositions for postulates and for theorems. Order is inherent in the deductive system form, but the particular order is accidental to the particular system. And for this the term "postulate" is meant to stand. It is a repudiation of Aristotelianism and a revival of Platonism.

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REVIEWS AND ABSTRACTS OF LITERATURE

Proceedings of the Aristotelian Society. 1910–1911. N. S., Vol. XI. Williams and Norgate.

As usual, the papers read before the Aristotelian Society, during its official year last completed, command the attention of every reader who would keep abreast of the philosophical times. With a few exceptions, the ten topics discussed and the opinions expressed about them are so important that a critical survey of them would greatly exceed the reviewer's proper bounds.

"Self as Subject and as Person," by S. Alexander, is an ingenious, pretty accurate, but not quite convincing analysis which aims to show "that the subject never is a presentation (or object), that the body of course is, and that the person (which is a combination of the former two) is partly presentation and partly not." There is, for Alexander, no pure ego to which objects are presented; the only ego is an experience, and this is not an object, but a bodily activity. (The term "object" here means of course any entity generically like a percept.) Different things require different actions in order to be cognized; and so, just as there is one type of behavior for knowing color, so too there is a distinct action for knowing one's "self." No peculiar complex of objects, no mere relation between past, present, and future things, constitute the material of this knowing. On the contrary, "just as the percept, the memory, the forecast of an external object as in the present, the past, or the future,